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Modeling golden section in plants

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Abstract

Plants are complex structures, changing their shapes in response to environmental factors such as sunlight, water and neighboring plants. However, some mathematical rules can be found in their growth patterns, one of which is the golden section. The golden section can be observed in branching systems, phyllotaxis, flowers and seeds, and often the spiral arrangement of plant organs. In this study, tree, flower and fruit models have been generated by using the corresponding golden section characteristics, resulting in more natural patterns. Furthermore, the golden section can be found in the bifurcate angles of trees and lobed leaves, extending the golden section theory.

Keywords: Golden section; Plant morphology; Plant modeling

1. Introduction

Plants are a major source of beauty in our natural surroundings. Different plants have different structures in terms of branches, leaves and flowers. Nevertheless, we can find some interesting and pervasive mathematical rules in them. Although trees and bushes differ in shape, their ratio of length to width is close to the golden section. In some plant stems, the divergence angle between two adjacent leaves approximates 137.28°. This is the central angle forming two radii, and if we divide the circumference into two parts, the ratio is 1:0.618. This angle promotes adequate ventilation of the plants and is the optimal arrangement for light absorbance. Numbers of leaves, petals and fruiting bodies in some plants follow the sequence of Fibonacci numbers. For example, spiral phyllotaxis or the layout of seeds in a pine cone reflects the Fibonacci numbers.

Furthermore, we also find that angles of plant bifurcation are not random. For most bushes or grasses such as bamboo, the bifurcation angle is 34.4°. For trees, poplar is 34.4°, and peach is 55.6°. These angles (55.6° and 34.4°) are the golden section of 90°. The golden section also exists in some lobed leaves, the lobe angles of Korean arborvitae and cypress are both 34.4°. The ratio between two pine needles is 0.618, as well as the ratio of leaf venation.

This magical proportion is almost omnipresent, and is an essential feature of plants. It has great significance in biology and is also a common phenomenon within the growth and evolutionary processes of plants.

The main aim of this study was to use these rules of plant morphology to generate a variety of different plant models. The results obtained provided more natural models of plant form and growth.

2. Related work

2.1. Plant morphology

The discipline of plant morphology focuses on the investigation and description of plant structural features and organization. It employs a variety of methods including direct observations, comparative studies and experimental

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manipulations. Among these, the methodologies of comparative morphology are particularly well suited for using natural variation to elucidate commonalities and convergences. In the past, people thought that there was no relationship between the morphological structure of plants and mathematics. Now, we find that our understanding of the golden section is increasingly expanded by studying nature. A developing trend in plant morphology is the mathematical reconstruction of plant structures as a means of understanding the mathematics underlying plant shape.

2.2. Plants modeling

Modeling plants is challenging, whether for computer games or garden design, because of the complexity of plant structures. More and more botanists and computer experts are working in this field. Within computer graphics, these studies can be roughly classified as L-system, fractal, particle system and/or reference axis techniques.

The fractal method uses a self-similarity function to generate the structure or shape of plants. Oppenheimer [1] proposed a fractal plant model, which allows users to input the branching angle, the number of branches in each stem and the size ratio between branches and main stems. Lindenmayer [2,3] proposed the formalism of L-systems as an original method for plant modeling, which was later developed by Prusinkiewicz et al. [4–9]. From this point on, plant modeling has become an important area of interdisciplinary research. Reeves and Blau [10] rendered complex trees using a set of small disks called particle systems to represent the leaves. This is a good way to draw grasses and forests. De Reffye [11] presented a reference axis technique model based on the birth and death of growing buds. In this the user controls the generation of plants using various parameters. Other work has attempted to reconstruct 3D tree geometry from 2D sketches. Okabe et al. [12] presented a system for designing 3D models of botanical trees quickly and easily using freehand sketches and additional example-based editing operations.

3. Golden section in plants

Differing environments and geographic locations make plant shapes look distinct. However, we can also find some rules here.

3.1. Golden section in branching

Plant branch structure is complex, and there is a large diversity of forms among different species. Nevertheless, plant branching systems can be defined as two major kinds:

(1) Single-axis patterns: such as poplar or pine trees. Here the apical meristematic tissue retains energy, which results in the main stem being of a higher status with respect to growth, while the lateral branches are inferior. This makes the plant shape high and straight (see Fig. 1a). Mature trees have six or more branches and growth direction, the angle between the two neighbors is about 135° and the angle between the main stem and each branch is close to 34.4° which is the golden section of 90° ($(90 - 33.4)/90 = 0.618$). Within 20 poplars, for every 100 branches, 79 branches satisfy this rule. This angle may be advantageous for leaves to absorb light and optimize the rate of photosynthesis.

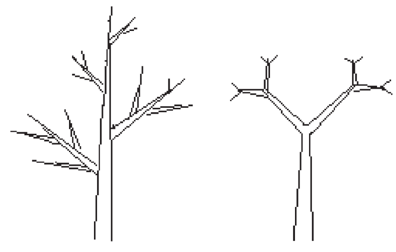

Fig. 1. The ramification patterns. (a) Single-axis pattern. (b) Gather-axis pattern.

(2) Gather-axis pattern: such as camphor trees and pear. Here the terminal bud of the branch dies within a certain time period, and then the lateral buds displace its role, forming strong side scions off the main axis. It does not have obvious main branches but forms many main stems causing the plant to have a sphere-like shape (see Fig. 1(b)). A mature tree has 2–5 branches, and the angle between branches and the main stem is not random, but is generally 55.6°. In 20 camphor trees, for every 100 branches, 58 branches are near to this angle.

3.2. Golden section in leaves

Leaves vary markedly in the form of the leaf blade, lobes, edge and venation. By careful observation and sampling, we have discovered many phenomena related to the golden section. In Korean arborvitae, each leaf has six vein branches (in a few of them there are 5 branches) and the scale-like shape is derived from the branches which create a flat surface. The leaf has two growth directions; the angle between the main venation and branches is about 34.4°. Among 100 Korean arborvitae leaves, 92 leaves satisfy this golden section. In Chinese pine, needles are grouped in twos. One is short, the other is long and the ratio of the two needles is 0.618. The angle between needles and the stem is near to 55.6°. Among 100 needle pairs, 84 pairs satisfy this golden section. Furthermore, for some lobed leaves such as maple and phoenix trees, the lobe angle is sometimes about 55.6°. Blade leaves such as those of poplar and camphor trees have an ellipse shape with a ratio between the short axis and the long axis of about 0.618.

The beautiful arrangements of leaves in some plants, called phyllotaxis, obey a number of subtle mathematical relationships. Surprisingly, these numbers are consecutive Fibonacci numbers (see Fig. 2). The ratios of alternate Fibonacci numbers are given by the convergents to Φ^{-2},

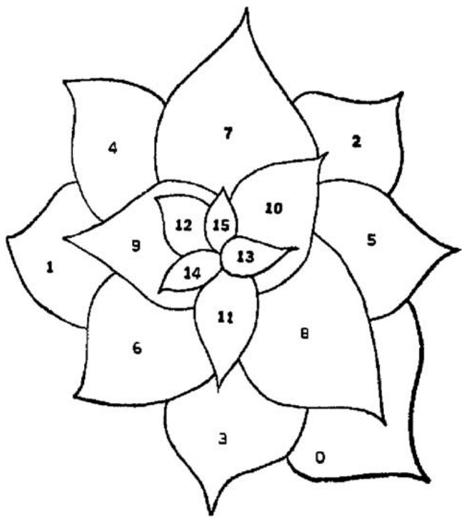

Fig. 2. Phyllotaxis.

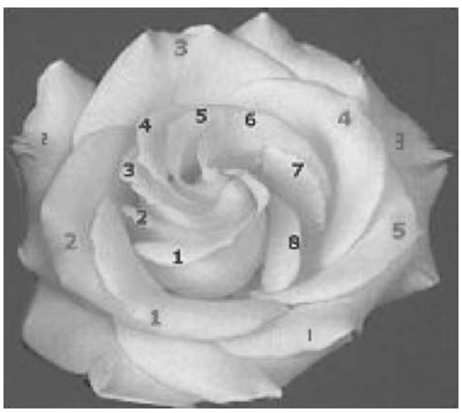

Fig. 3. The phyllotaxis of Chinese rose.

where Φ is the golden section. This closely fits the fraction of a turn between successive leaves on the stalk of certain plants: 1/2 for elm and linden, 1/3 for beech and hazel, 2/5 for oak and apple, 3/8 for poplar and rose, 5/13 for willow and almond and so on [13,14].

Some flowers have a large number of petals such as the Chinese rose. The petals are often arranged in spiral patterns with delamination. From inside to outside, the numbers of petals are 21, 13, 8, 5, and 3 based on their size (see Fig. 3). These numbers are adjacent Fibonacci numbers.

We can discover the golden section in fruits in many forms. For instance, the florets in the head of a sunflower form two oppositely directed spirals: 55 of them are clockwise and 34 are counterclockwise. A similar phenomenon occurs in daisies, pineapples, pinecones, cauliflowers, and so on.

4. Modeling method

Definition 1. Define the transformation $W:R^3 \to R^3$ as follows:

$$W\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & k \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} u \\ v \\ r \end{bmatrix}$$

where a,b,c,d,e,f,g,h,k,u,v,r are real numbers and W is a three dimensional affine transformation. When $X \in R^3$, the above expression may be written as: $W(X) = AX + t$, where $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & k \end{bmatrix}$ and $t = \begin{bmatrix} u \\ v \\ r \end{bmatrix}$ is a translation transformation.

A is the blend of the following four kinds of affine transformations, $\begin{bmatrix} sx & 0 & 0 \\ 0 & sy & 0 \\ 0 & 0 & sz \end{bmatrix}$ is a scale transformation; $\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{bmatrix}$, $\begin{bmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{bmatrix}$ and $\begin{bmatrix} \cos\gamma & \sin\gamma & 0 \\ -\sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{bmatrix}$ are rotation transformation using the x-, y-, and z-axis, respectively.

The following coordinate transformation allows shifting to a new local coordinate space:

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\beta\cos\gamma & \cos\beta\sin\gamma & \sin\beta & u \\ \sin\alpha\sin\beta\cos\gamma - \sin\gamma\cos\alpha & \sin\alpha\sin\beta\sin\gamma - \cos\gamma\cos\alpha & -\cos\beta\sin\gamma & v \\ -\sin\alpha\sin\beta\cos\gamma - \sin\gamma\sin\alpha & -\sin\gamma\sin\beta\cos\alpha + \sin\alpha\cos\gamma & \cos\beta\cos\alpha & r \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

3.3. Golden section in flowers and fruits

Lilies, irises, and trillium have three petals; columbines, buttercups, larkspur, and wild rose have five petals; delphiniums, bloodroot, and cosmos have eight petals; corn marigolds have 13 petals; asters have 21 petals; and daisies have 34, 55, or 89 petals. All of these are Fibonacci numbers.

While the results are generated recursively, and all previous local coordinate transformations are accumulated. Although the shape of natural trees is complex and various, the final results of these models are more regular. To make the graphic look more natural, we added a control function for which the parameters could be changed in the iterative process. The function is shown as:

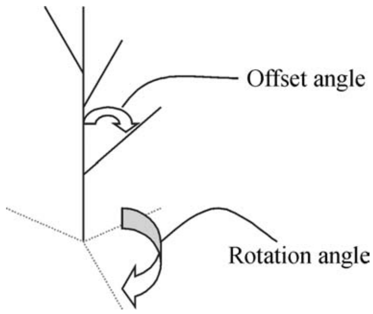

Fig. 4. The offset angle and rotation angle.

$$\text{Modify}() = \begin{cases} \text{high'} = (1 - \text{randum}(0, 0.1) \cdot \text{branch}) \\ \quad \cdot \text{high} \\ \text{radius'} = \text{randum}(0, 0.2) \cdot (\text{branch} + 1) \\ \quad \cdot \text{radius} \\ \text{osangle'} = \text{osangle} + 5 \cdot \text{branch} \\ \text{roangle'} = \text{roangle} + 5 \cdot \text{branch} \\ \text{curve'} = \text{curve} + \text{randum}(-0.05, 0.05) \\ \quad \cdot \text{branch} \end{cases}$$

In the above function, high is the height of branching, radius is the radius of segment, osangle is offset angle, roangle is rotation angle (see Fig. 4), curve is the curve factor and branch is the frequency of branching. Thus the minor branches are different to the major branches they stem from in each iteration. This makes the resultant model look more realistic.

Fig. 5. Poplar leaves.

Table 1
Codes of poplar IFS.

n	sx	sy	sz	α	β	γ	u	v	r
1	0.90	0.90	0.90	0	34.4	0	0	0	0
2	0.95	0.95	0.95	137.5	34.4	0	0	0.05	0
3	0.90	0.90	0.90	275	34.4	0	0	0.10	0
4	0.90	0.90	0.90	52.5	34.4	0	0	0.15	0
5	0.85	0.85	0.85	190	34.4	0	0	0.20	0
6	0.38	0.38	0.38	327.5	34.4	0	0	0.25	0

5. Results

Plant leaves exhibit great variety; and the vein and border patterns are complex. This rich information is difficult to simulate. In this study, we have extracted alpha masks from leaf photographs using the algorithm of Ruzon and Tomasi [15]. This algorithm requires user intervention, in which the zones of background and foreground need to be specified. Billboard textures were used to generate the leaves (see Fig. 5). For creating similar petals or fruit squamas, we used NURBS surfaces to simulate them and then used an iteration algorithm to generate different models.

5.1. Poplar

Poplar branching structure follows the single-axis pattern. There are more than six main branches at each level and each branch has six main growth directions. The angle between two neighboring branches is approximately 135°; the angle between branches and the main stem is about 34.4°. According to these morphological properties of poplars, we produced an iteration function system (IFS) for poplar with six affine transformation matrices (see Table 1). The results are shown in Fig. 6.

5.2. Peach

The peach branching structure follows the multi-axis pattern. The terminal branches stop growing after a period of time, and sub-branches replace them as the main growing tips. Peach has 2–4 fixed branches, and the angle

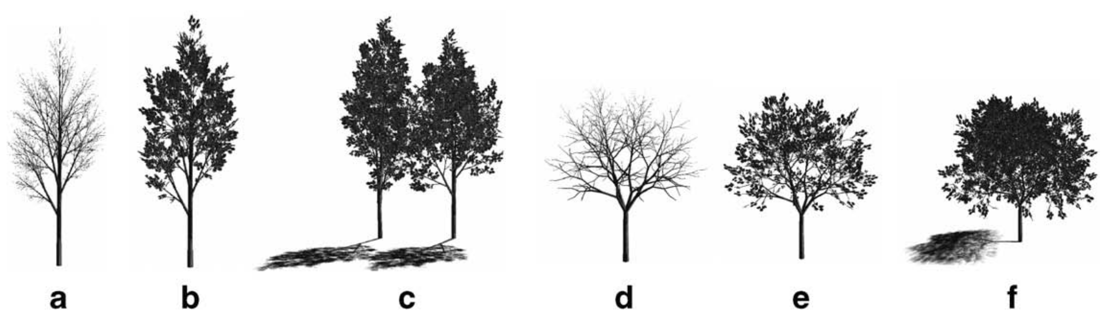

Fig. 6. The models are more natural using golden section rules. (a) The poplar branch structure with six affine transformation matrices. (b) The poplar with leaves, the phyllotaxis is 3/8, the adjacent leaf angle is $2\pi \times 3/8 = 135°$. (c) Two different poplars changed using the modification function. (d) The peach branch structure with four affine transformation matrices. (e) The peach with leaves, the phyllotaxis is 2/5, the adjacent leaf angle is $2\pi \times 2/5 = 144°$. (f) The shape is changed by adding the leaf density with the modification function.

Table 2
Codes of peach IFS.

n	sx	sy	sz	α	β	γ	u	v	r
1	0.62	0.62	0.62	0	55.6	0	0	1.0	0
2	0.62	0.62	0.62	137.5	55.6	0	0	1.0	0
3	0.62	0.62	0.62	275	55.6	0	0	1.0	0
4	0.62	0.62	0.62	52.5	55.6	0	0	1.0	0

between a branch and the main stem is not random but generally 55.6°. According to these morphological properties of peach, we produced an iteration function system with four affine transformation matrices (see Table 2). The results are shown in Fig. 6.

5.3. Leaf

The leaves of Korean arborvitae have six vein branches (and a few of them have 5 branches). These branches create a flat surface. The leaves have two growth directions, and the angle between the main venation and the branches is about 34.4°. According to these morphological properties of Korean arborvitae, we produced an iteration function system with six affine transformation matrices (see Table 3). The results are shown in Fig. 7.

5.4. Flower and fruit

Daisies and the Chinese rose arrange their petals in spiral patterns (see Fig. 8). Firstly, we assume that the petals grow within the same circle. Then the angle of adjacent petals of the same layer is $\varphi = \frac{2\pi}{n_i}$, where n_i is the petal number at the i-th layer. So, we can compute the spiral angle as follows: for daisies (see Fig. 9), they are 10.6° and 17.1°, and for Chinese rose (see Fig. 10), they are 27.7°, 45°, 72° and so on.

Table 3
Codes of Korea arborvitae IFS.

n	sx	sy	sz	α	β	γ	u	v	r
1	0.50	0.50	0.50	−34	0	0	0	0.50	0
2	0.45	0.45	0.45	34	0	0	−0.08	0.90	0
3	0.40	0.40	0.40	−34	0	0	0.10	1.20	0
4	0.30	0.30	0.30	34	0	0	0.10	1.40	0
5	0.25	0.25	0.25	−34	0	0	0.30	1.70	0
6	0.20	0.20	0.20	34	0	0	0.31	1.80	0

Fig. 7. Korea arborvitae.

Fig. 8. Petals arranged in spiral pattern.

Fig. 9. Daisy.

Fig. 10. Chinese rose.

To arrange N units in a sphere following golden section rules (see Fig. 9a), we assume that the N units are around the z-axis with the angles $\Phi i = i \times \mathrm{d}\Phi$, and for the i-th unit that:

$$\mathrm{d}\Phi = 2\pi/((1+\sqrt{5})/2) = 137.5^\circ$$

137.5° is the golden section angle. The problem is how to compute θ_p. Firstly, the area of the gray region was computed (see Fig. 11a), $S = 2\pi Rh$, $h = R(\cos\ \theta_1 - \cos\ \theta_2)$, where S_0 is the area of one unit. If we placed N units in a sphere, the radius of the sphere is:

$R = \sqrt{NS_0/2\pi(\cos\theta_1 - \cos\theta_2)}$. θ_p can then be obtained using:

$$\sin\ \theta_p = \frac{R - R(1-\cos\ \theta_1) - \frac{S_0 h}{S}}{R} = \cos\ \theta_1 - \frac{S_0}{S}(\cos\ \theta_1 - \cos\ \theta_2)$$

Finally we used this method to generate the patterns found in pine cones and sunflowers (see Fig. 11b and c).

6. Conclusions and future work

In this study, we have constructed a variety of plant models using gross morphological data and various algorithms, and examined the golden section in plant forms.

These observations of the role of the golden section in plants are summarized in Table 4.

The above statistics confirm the prevalence of the golden section in plants. Possibly, these relationships are advantageous for plant growth and have developed over an evolutionary time scale.

Many more rules for the golden section in plants remain to be described in the future. Plant models require more interactive interfaces allowing modification of features such

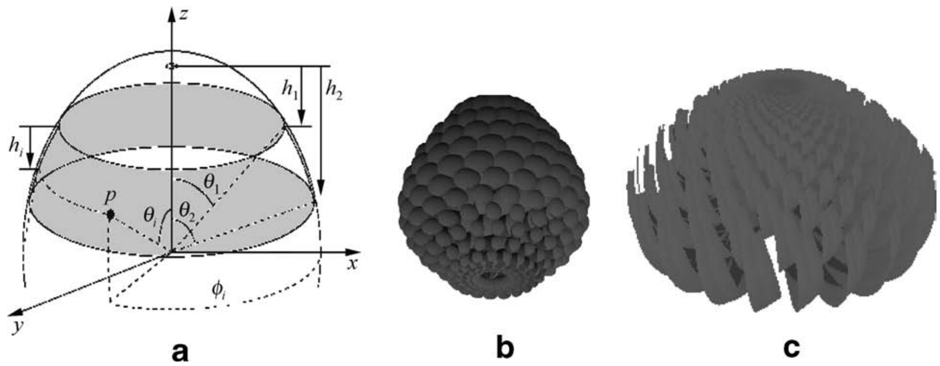

Fig. 11. The golden section on a sphere (a), pinecone pattern (b), and a sunflower pattern (c).

Table 4
The statistics of golden section in plants.

Plants	Golden section
Branches	Bifurcation angle and bifurcation point
Leaves	Phyllotaxis, ratios between the short axis and the long axis of blade leaves and the bifurcation angle of the lobed leaves
Flowers	Petal numbers and mode of arrangement
Fruits	Ftuit numbers and mode of arrangement

as leaf shape, branch structure, as well as free-form deformation of stems without affecting the leaves. In this paper, collisions of plant parts (e.g. branches, leaves) have not been considered. Neither have the effects of water supply and intensity of sunshine been included. These features of growth and ambient factors remain to be considered in future research.

Acknowledgments

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